Diagonal Arguments and Infinite Sets

David Wyde

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Abstract

The diagonal arguments by Cantor and Church prove that it is impossible to enumerate infinite sets.

1 Introduction

A diagonal argument shows that any finite subset of an infinite set is missing a value. Cantor used this to prove that infinite sets can have different sizes, and Church made a similar case to show the impossibility of telling if a formula in the λ-calculus can be reduced to a normal form.

2 Cantor's Diagonal Proof

Cantor introduced the diagonal argument to show that there is an infinite set that cannot be arranged in a one-to-one correspondence with the natural numbers 1, 2, 3, ... [1].

The outline of Cantor's diagonal argument is:

1. Make a list of every infinite binary string.
2. Create a new binary string.
3. Show that the string in Step 2 is not in the list from Step 1.
4. Notice that Step 3 contradicts the claim that Step 1 contains every binary string.
5. Prove that there is an infinite set that has no one-to-one correspondence with the natural numbers.

Cantor's argument examines an infinite set of binary strings. Each string has an infinite number of bits; this paper will use 0's and 1's. Cantor's proof lists every possible string in a table. Cantor argued that there is always a binary string that is not in the table.

The partial table below contains three values: 000, 111, and 010. Each of these strings extends infinitely to the right, and the table contains an infinite number of strings.

0	0	0	...
1	**1**	1	...
0	1	**0**	...

...

In this example, the diagonal value is 010...: the first bit of the first line, the second bit of the second line, and so on. Step 2 creates a new string by flipping every bit of the diagonal value: 0's become 1's and vice versa. This new string, 101, has at least one bit different for every row, so it is guaranteed to not be in the table.

This absence justifies Step 3's assertion that the new value did not exist at Step 1, which proves Step 4's claim that the list in Step 1 is incomplete.

Step 5 is a statement about the sizes of sets. Cantor argued that Steps 1–4 prove that the infinite set of real numbers is larger than the infinite set of natural numbers.

3 Church's Undecidability Proof

Church used a diagonal argument when analyzing Hilbert's decision problem[2]. His argument focuses on the λ-calculus, a way to define functions.

The λ-calculus performs computations by applying rules to a formula. A formula with no function calls in its body is in normal form. This serves as a standard, reduced version of equivalent formulas.

One formula can be turned into another through a series of substitutions. Church gave three ways to transform a formula: rename a variable, replace a function call with the result of applying it, and replace a value with a function call that would produce it.

Given two formulas $\boldsymbol{A}$ and $\boldsymbol{B}$, it may or may not be possible to apply a sequence of the above three transformations to convert $\boldsymbol{A}$ into $\boldsymbol{B}$. This is an unbounded search problem: if the task is impossible for a given $\boldsymbol{A}$ and $\boldsymbol{B}$, the naive approach of trying one transformation after another will never terminate.

Church's proof parallels Cantor's diagonal argument:

1. Make a list of every formula that has a normal form.
2. Create a new formula.
3. Show that the formula in Step 2 is not in the list from Step 1.
4. Notice that Step 3 contradicts the claim that Step 1 contains every formula that has a normal form.
5. Prove that there can be no algorithm to determine if a formula has a normal form.

The diagonalization occurs in Step 3, which defines a new formula that has a different output than any formula listed in Step 2.

Every formula has an integer identifier. Church's diagonalized formula takes an integer argument `n`. If the formula $\boldsymbol{n}$ that corresponds to `n` can be transformed to a formula in normal form $\boldsymbol{m}$, the diagonalized formula returns $\boldsymbol{m}$'s identifier plus one. Otherwise, the diagonalized formula returns `1` for that value of `n`.

The new formula cannot be converted to any existing formula with a normal form. As in Cantor's proof, Church's diagonalization leads to a value in Step 2 that is not listed in Step 1. Again following Cantor, Church argued that this missing element supports the conclusion in Step 5.

4 Conclusion

In both of the proofs outlined above, Step 5 does not follow from Steps 1–4. Cantor's and Church's diagonal arguments disprove the assumption that it is possible to list every element of an infinite set, at which point (Step 4) both proofs should end.

The modern understanding of infinity, based on Cantor's work, leads to paradoxical results. Banach and Tarski proved that one sphere can be rearranged into two separate spheres, each the same size as the original, through an algorithm that breaks the original shape into an infinite number of points[3].

An alternative interpretation is that infinity means there are always values beyond the current one: it is not possible to reach infinity. This debate dates at least to Aristotle[4], and perhaps it will continue[5].

References

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[2] A. Church. "An Unsolvable Problem of Elementary Number Theory," *American Journal of Mathematics*, vol. 58, no. 2, pp. 345–363, 1936.

[3] R. French, "The Banach-Tarski Theorem," *The Mathematical Intelligencer*, vol. 10, pp. 21–28, 1988.

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